Calculus II - Day 12

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Goals for today:

- "Reverse the chain rule" to integrate by substitution when functions appear.
- Change the limits of integration to compute definite integrals.

Reminder:

View your graded midterm in Gradescope and bring any questions to practicum this week

Recall: The Chain Rule

$$\frac{d}{dx} f(g(x)) = g'(x) \cdot f'(g(x))$$

Ex. $\frac{d}{dx}\cos(\ln(x))$

$$g(x) = \ln(x), \quad f(x) = \cos(x)$$

$$g'(x) = \frac{1}{x}, \quad f'(x) = -\sin(x)$$

$$\frac{1}{x} \cdot (-\sin(\ln(x))) = -\frac{\sin(\ln(x))}{x}$$

Another way: $y = f(u)$ where $u = g(x)$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Ex. $\frac{d}{dx}\cos(\ln(x))$

$$u = \ln(x), \quad y = \cos(u)$$

$$\frac{du}{dx} = \frac{1}{x}, \quad \frac{dy}{du} = -\sin(u)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = -\sin(u) \cdot \frac{1}{x} = -\frac{\sin(\ln(x))}{x}$$

How do we do this for integrals?

The u-substitution rule:

$$\int f'(g(x)) \cdot g'(x)\,dx = f(g(x)) + C$$

How to use: find an "inside function" $u = g(x)$ whose derivative $u' = g'(x)$ appears in the integrand.

$$\int 2x\sqrt{1+x^2}\,dx$$

Let $u = 1 + x^2$

$$\frac{du}{dx} = 2x, \quad du = 2x\,dx$$

$$\int \underline{\sqrt{1+x^2}} \cdot \underline{2x\,dx} = \int \underline{\sqrt{u}} \cdot \underline{du} = \int u^{1/2}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1+x^2)^{3/2} + C$$

Check to see if this is correct by differentiating:

$$\frac{d}{dx}\left[\frac{2}{3}(1+x^2)^{3/2} + C\right] = \frac{2}{3} \cdot 2x \cdot \frac{3}{2}(1+x^2)^{1/2} = 2x\sqrt{1+x^2}$$

Ex.

$$\int (3x^2+4)^4 \cdot 6x\,dx$$

Let $u = 3x^2 + 4$, so $du = 6x\,dx$

$$= \int u^4\,du = \frac{1}{5}u^5 + C = \frac{1}{5}(3x^2+4)^5 + C$$

These are examples of "perfect substitution."

Ex.

$$\int 2e^{10x}\,dx$$

Let $u = 10x$, so $du = 10\,dx$

$$= \int e^{10x} \cdot 2\,dx = \int e^u \cdot \frac{1}{5}\,du$$

$$= \frac{1}{5}e^u + C = \frac{1}{5}e^{10x} + C$$

"imperfect substitution"

Ex.

$$\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\,dx$$

Let $u = \sqrt{x}$, so $du = \frac{1}{2\sqrt{x}}\,dx$

$$= \int \cos(\sqrt{x}) \cdot \frac{1}{\sqrt{x}}\, dx = \int \cos(u) \cdot 2\, du$$

$$= 2\sin(u) + C = 2\sin(\sqrt{x}) + C$$

The professor asks the class to try the following substitutions. Provided are sample solutions: a)

$$\int 2\sin^3(x)\cos(x)\, dx$$

Let $u = \sin(x)$, so $du = \cos(x)\, dx$

$$= \int 2u^3\, du = \frac{2}{4}u^4 + C = \frac{1}{2}\sin^4(x) + C$$

b)

$$\int \frac{x^2}{x^3+7}\, dx$$

Let $u = x^3 + 7$, so $du = 3x^2\, dx$, hence $\frac{du}{3} = x^2\, dx$

$$= \int \frac{1}{3} \cdot \frac{1}{u}\, du = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|x^3+7| + C$$

Ex. A variation

$$\int \frac{x}{\sqrt{x-1}}\, dx$$

Let $u = x - 1$, so $du = dx$, and $x = u + 1$

$$= \int \frac{(u+1)}{\sqrt{u}}\, du = \int \left(\frac{u}{\sqrt{u}} + \frac{1}{\sqrt{u}}\right) du$$

$$= \int \left(u^{1/2} + u^{-1/2}\right) du = \frac{2}{3}u^{3/2} + 2u^{1/2} + C$$

$$= \frac{2}{3}(x-1)^{3/2} + 2(x-1)^{1/2} + C$$

What if we compute a definite integral instead?

$$\int_e^{e^3} \frac{1}{x\ln(x)}\, dx$$

Let $u = \ln(x)$, so $du = \frac{1}{x}\, dx$

We have to change the limits of integration to reflect that the variable in the new integral is u, not x.

- When $x = e$, $u = \ln(e) = 1$ - When $x = e^3$, $u = \ln(e^3) = 3$

Thus, the integral becomes:

$$\int_1^3 \frac{1}{u}\, du$$

Now, integrate:

$$= \ln|u|\Big|_1^3 = \ln(3) - \ln(1) = \ln(3)$$

So the result is $\ln(3)$.

What if we instead compute the indefinite integral first?

$$\int \frac{1}{x\ln(x)}\,dx$$

Let $u = \ln(x)$, so $du = \frac{1}{x}\,dx$

$$= \int \frac{1}{u}\,du = \ln|u| + C = \ln|\ln(x)| + C$$

That means:

$$\int_e^{e^3} \frac{1}{x\ln(x)}\,dx = (\ln|\ln(x)| + C)\Big|_e^{e^3}$$

$$= \ln|\ln(e^3)| - \ln|\ln(e)|$$

$$= \ln|3| - \ln|1| = \ln(3)$$

(same as earlier)

Ex.

$$\int_0^2 \frac{1}{(2x+2)^2}\,dx$$

Let $u = 2x + 2$, so $du = 2\,dx$, which means $dx = \frac{1}{2}\,du$.

The professor notes that the function must be continuous over the interval of integration, so if the domain included $x = -1$, the integral would be invalid.

Now, change the limits of integration:

$$u(0) = 2 \times 0 + 2 = 2$$

$$u(2) = 2 \times 2 + 2 = 6$$

Substitute u and du into the integral:

$$\int_0^2 \frac{1}{(2x+2)^2}\,dx = \int_2^6 \frac{1}{u^2} \cdot \frac{1}{2}\,du$$

Now integrate:

$$= -\frac{1}{2}u^{-1}\Big|_2^6$$

Substitute the limits:

$$= -\frac{1}{2}\left(\frac{1}{6} - \frac{1}{2}\right)$$

Simplify:

$$= -\frac{1}{2} \cdot -\frac{1}{3} = \frac{1}{6}$$

Ex.

$$\int_0^1 xe^{-x^2}\,dx$$

Let $u = -x^2$, so $du = -2x\,dx$.
Change the limits of integration:

$$u(0) = 0$$

$$u(1) = -(1)^2 = -1$$

Substitute into the integral:

$$\int_0^1 xe^{-x^2}\,dx = \int_0^{-1} -\frac{1}{2}e^u\,du$$

Now integrate:

$$= -\frac{1}{2}e^u\Big|_0^{-1}$$

Substitute the limits:

$$= -\frac{1}{2}\left(\frac{1}{e} - 1\right)$$

Simplify:

$$= \frac{1}{2} - \frac{1}{2e}$$

More example questions:

a)

$$\int_0^{\frac{\pi}{4}} 2\sec^2(x)\tan(x)\,dx$$

Let $u = \tan(x)$, so $du = \sec^2(x)\,dx$
Change the limits of integration:

$$\tan(0) = 0, \quad \tan\left(\frac{\pi}{4}\right) = 1$$

Substitute into the integral:

$$\int_0^{\frac{\pi}{4}} 2\sec^2(x)\tan(x)\,dx = \int_0^1 2u\,du$$

Now integrate:

$$= u^2\Big|_0^1 = 1^2 - 0^2 = 1$$

b)

$$\int_0^1 \frac{x}{1+x^4}\,dx$$

Let $u = x^2$, so $du = 2x\,dx$

Change the limits of integration:

$$u(0) = 0^2 = 0, \quad u(1) = 1^2 = 1$$

Substitute into the integral:

$$\int_0^1 \frac{x}{1+x^4}\,dx = \int_0^1 \frac{1}{2}\frac{du}{1+u^2}$$

Now integrate:

$$= \frac{1}{2}\arctan(u)\Big|_0^1$$

Substitute the limits:

$$= \frac{1}{2}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{8}$$

Note:

For part (a), you can also choose $u = \sec(x)$:

$$du = \sec(x)\tan(x)\,dx$$

$$\sec(0) = 1, \quad \sec\left(\frac{\pi}{4}\right) = \sqrt{2}$$

$$\int_0^{\pi/4} 2\sec^2(x)\tan(x)\,dx = \int_0^{\pi/4} 2\sec(x)\cdot\sec(x)\tan(x)\,dx$$

$$= \int_1^{\sqrt{2}} 2u\,du$$

$$= u^2\Big|_1^{\sqrt{2}} = (\sqrt{2})^2 - (1)^2 = 2 - 1 = 1$$